The Propagation of Star-Shaped Brittle Cracks

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The paper studies the dynamical propagation of star-shaped cracks symmetrically arranged in an elastic thin plate, subjected to the action of instantly applied, comprehensively (uniformly) stretching stresses, which implies a self-similar problem with homogeneous stresses and velocities of particles. Occurrence of such motion patterns is established through experiments. By using the Smirnov–Sobolev functional-invariant solutions method and a careful choice of mappings, the problem is reduced to some boundary value problem of the theory of complex variable functions, and exact analytic solution of the original problem, including a closed-form solution for important stress intensity coefficient near the end of the crack, is derived. We also establish a fundamental theoretical limit imposed on the number of cracks—there has to be at least three cracks. [DOI: 10.1115/1.4000900]

Dedicated to the memory of Yu. A. Amenzade.

1 Introduction

The propagation of a rectilinear (straight) crack in an elastic plane, assuming the ideal model of brittle failure, is the most commonly considered problem of dynamical mechanics. Furthermore, it is assumed that unlimited stresses occur at the tip of the crack and are described by the intensity factor. In this direction, the problem of the propagation of cracks was considered by many authors and a great deal of them is presented in the monograph [1].

The present paper, by its very formulation, significantly differs from the ideal case: It allows for an arbitrary number of multiple cracks propagating in different directions as opposed to a single crack. These cracks originate from the same point and are symmetrically positioned around the point of origin with period $2\pi/n$, where n is the number of propagating cracks. A number of scientific experiments were carried out in the Wave Dynamics Laboratory of Institute of Mathematics and Mechanics of National Academy of Azerbaijan that confirm the existence of such motion patterns (see Fig. 1)

It is evident that solving this problem as an initial-boundary problem (to determine the intensity factor) requires development of a completely new methodology, as this is an unstationary (dynamic) problem of elastic dynamics for a region with nonclassical boundary.

On the other hand, the current problem can also be considered as a generalized case of the so-called Broberg's problem [2], which considers a self-similar problem of propagation of a rectilinear crack in thin uniformly stretched membranes. Much simpler solution of Broberg's problem than the original one is presented in Refs. [3,4] that uses functional-invariant solutions of Smirnov–Sobolev [5]. That method could only be applied to rectilinear cracks (one or several), propagating and located on the same line.

Since the problem considered in this paper assumes neighboring cracks are separated by an angle of $2\pi/n$, we had to use a completely novel approach that combines a number of different methods and tools of the theory of mathematical physics and the theory of functions of complex variables. In particular, invariant solutions of Smirnov–Sobolev described in Ref. [5] enable reducing a solution of a two-dimensional wave equation to some boundary value problem in the corresponding complex plane. However, existence of two types of motion (potential and solenoidal) implies, under this method, utilizing two different complex planes, which make up the main difficulty in constructing a solution. Nevertheless, complex representation turns out to be the only possibility for solving a majority of the problems of not only static, but also dynamic plane theory of elasticity.

2 Problem Statement and Complex Representation of the Solution

Consider an infinite homogeneous isotropic elastic thin plate subjected to the action of instantly applied, comprehensively (uniformly) stretching stresses. Assume that at the moment $t=0$ in the origin of coordinates, a number of cracks appear and stretch with constant velocity in different directions. Existence of such a motion pattern is confirmed by numerous experiments shown in Fig. 1 (as carried out in "Wave Dynamics" Laboratory of IMM of NAS of Azerbaijan). Please note that sometimes, imprecisions in the experiments might lead to the violation of the symmetric problem formulation, and hence, the cracks might not be symmetrically positioned.

The cracks are arranged symmetrically around the origin with period $2\pi/n$, where n is the number of cracks (Fig. 2). The propagation velocity is the same for all the cracks and less than the propagation velocity of the Rayleigh cylindrical waves.

In the plane dynamic theory of elasticity, the displacement vector $\boldsymbol{U}$ may be represented as the sum of potential vector $\boldsymbol{U}_1$ and solenoidal vector $\boldsymbol{U}_2$

$$\boldsymbol{U}=\boldsymbol{U}_1+\boldsymbol{U}_2 \tag{1}$$

$$\text{rot}\ \boldsymbol{U}_1=0,\quad \text{div}\ \boldsymbol{U}_2=0,\quad \boldsymbol{U}_k=\{u_k,\upsilon_k\},\quad k=1,2 \tag{2}$$

By virtue of its statement, the problem is self-similar, and velocities and stresses are homogeneous functions of degree zero, i.e., $\dot{u}_k,\dot{\upsilon}_k,\sigma_{xx},\sigma_{yy},\sigma_{xy}$ may be represented as functions of $(x/at, y/at)$.

Taking into account the fact that Cartesian components of velocities (u_k,υ_k) (or stresses) satisfy wave equations, it is possible to represent the stress functions by the Smirnov–Sobolev functional-invariant solution [5, pp. 196–214]

$$\frac{\partial u_k}{\partial x}=\text{Re}\ U_k(z_k),\quad \frac{\partial \upsilon_k}{\partial x}=\text{Re}\ V_k(z_k),\quad k=1,2 \tag{3}$$

Here, $U_k(z_k)$ and $V_k(z_k)$ are analytical functions of complex variables

$$z_1=\frac{1-\sqrt{1-\rho^2}}{\rho}\cdot e^{i\theta},\quad z_2=\frac{\bar{b}-\sqrt{\bar{b}^2-\rho^2}}{\rho}\cdot e^{i\theta} \tag{4}$$

where $\rho=r/at$, $\bar{b}=b/a$, and r,θ are the polar coordinates, and a and b are the velocities of longitudinal and transverse waves, respectively.

From Eq. (3), we have

$$u_k=\text{Re}\int U_k(z_k)dx,\quad \upsilon_k=\text{Re}\int V_k(z_k)dx,\quad k=1,2 \tag{5}$$

Since rot $\bar{V}_1=0$ and div $\bar{V}_2=0$, we have

$$i\left(z_1-\frac{1}{z_1}\right)U_1'(z_1)=\left(z_1+\frac{1}{z_1}\right)V_1'(z_1)\quad \text{and}$$

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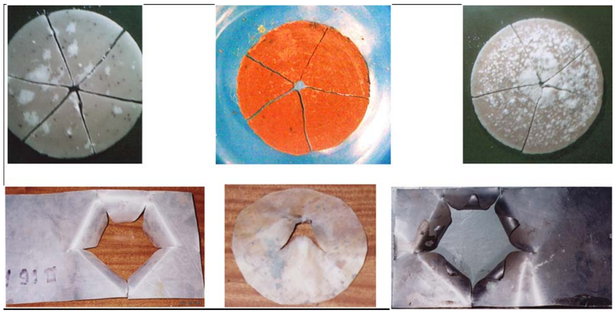

Fig. 1 Existence of star-shaped cracks experimentally verified

$$\left(z_2+\frac{1}{z_2}\right)U_2'(z_2)=-i\left(z_2-\frac{1}{z_2}\right)V_2'(z_2) \tag{6}$$

The solution of the problem in the polar coordinate θ will be periodic with period $2\pi/n$, and the main domain will have the form depicted in Fig. 2, where OC and OC' are borders of neighboring cracks, and AA_1A' and BB_1B' are the longitudinal and transverse wavefronts, respectively. Initial comprehensive stress state σ_0 holds ahead of the forefront AA'. The following boundary conditions hold

$$\sigma_\theta=\sigma_{r\theta}=0 \quad \text{on} \quad OC \text{ and } OC'$$

$$\sigma_{r\theta}=u_\theta=0 \quad \text{on} \quad CA \text{ and } C'A' \tag{7}$$

$$\sigma_r=\sigma_\theta=\sigma_0 \quad \text{on} \quad AA'$$

As can be seen from Eq. (7), the motion is symmetric in the main domain as well, and therefore, it suffices to investigate it in the domain AOA_1, where the condition $\sigma_{r\theta}=\dot{u}_\theta=0$ holds on the symmetry line OB_1A_1.

The choice of coordinate systems is shown in Fig. 2. The symmetry line OA_1 in this plane will have polar coordinates (r,α), where $\alpha=\pi/n$.

Using formulas (3), (4), and (6), as well as the formulas connecting Cartesian components of tensors of stresses and velocity vectors with their polar components, and following Refs. [3,4], we can express boundary condition (7) by the boundary values of analytical functions as

$$\frac{\sigma_{r\theta}}{\mu}=-2\,\mathrm{Re}\int\left\{\left[-1-\left(\frac{1-z_1^2}{1+z_1^2}\right)^2\right]\cos\theta\sin\theta-\frac{i(z_1^2-1)}{z_1^2+1}\cdot\cos 2\theta\right\}U_1'(z_1)dz_1-\mathrm{Re}\int\left\{\{2\sin 2\theta\}+\cos 2\theta\left[\frac{i(z_2^2-1)}{z_2^2+1}-\frac{z_2^2+1}{i(z_2^2-1)}\right]\right\}U_2'(z_2)dz_2 \tag{8}$$

$$\frac{\sigma_\theta}{\mu}=\mathrm{Re}\int\left\{\left[-\left(\frac{1}{\bar{b}^2}-2\right)\left(\frac{i(z_1^2-1)}{z_1^2+1}\right)^2+\frac{1}{\bar{b}^2}\right]\sin^2\theta+\left[\left(-\frac{1}{\bar{b}^2}-2\right)-\frac{1}{\bar{b}^2}\left(\frac{z_1^2-1}{z_1^2+1}\right)^2\right]\cos^2\theta-2\sin 2\theta\frac{i(z_1^2-1)}{z_1^2+1}\right\}U_1'dz_1+\mathrm{Re}\int\left\{-2\cos 2\theta-\sin 2\theta\left[\frac{i(z_2^2-1)}{z_2^2+1}-\frac{z_2^2+1}{i(z_2^2-1)}\right]\right\}U_2'(z_2)dz_2 \tag{9}$$

$$\frac{\dot{u}_\theta}{a}=-\mathrm{Re}\int\left[\frac{2z_1\sin\theta}{z_1^2+1}+\frac{2i(z_1^2-1)z_1\cos\theta}{(z_1^2+1)^2}\right]U_1'(z_1)dz_1-\mathrm{Re}\int\left[\frac{2\bar{b}\sin\theta}{z_2^2+1}z_2-\frac{2\bar{b}\cos\theta}{i(z_2^2-1)}z_2\right]U_2'(z_2)dz_2 \tag{10}$$

where μ is a Lame constant.

In Sec. 3, by using the representations derived here, we develop a method to arrive at the solution for functions V_1' and V_2', and consequently derive the solution for σ_θ and determine the stress intensity coefficient.

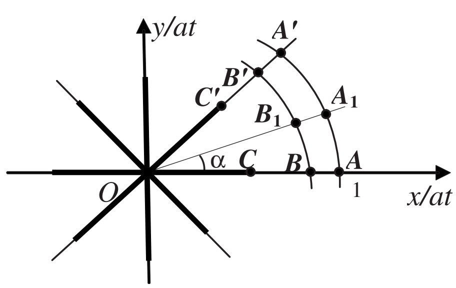

Fig. 2 Choice of coordinate system

3 Development of the Solution

Note that the conditions on the boundary of segment $\theta=\alpha$ enable us to separate the values of the functions U_1' and U_2' on this boundary. Specifically, from the conditions

$$\sigma_{r\theta}|_{\theta=\alpha}=\sigma_{r\theta}^1|_{\theta=\alpha}+\sigma_{r\theta}^2|_{\theta=\alpha}=0 \quad \text{and} \quad \dot{u}_\theta|_{\theta=\alpha}=\dot{u}_\theta^1|_{\theta=\alpha}+\dot{u}_\theta^2|_{\theta=\alpha}=0 \tag{11}$$

we get that

$$\dot{u}_\theta^1=0 \quad \text{and} \quad \dot{u}_\theta^2=0 \quad \text{on} \quad \theta=\alpha \tag{12}$$

Thus, the values of the different function arguments remain bounded only on the boundary of the segment for $\theta=0$, where, taking into account Eqs. (8) and (10), condition (7) takes the form

$$\frac{\sigma_\theta}{\mu}=\mathrm{Re}\int\left[\left(\frac{1}{\bar{b}^2}-2\right)-\frac{1}{\bar{b}^2}\left(\frac{z_1^2-1}{z_1^2+1}\right)^2\right]U_1'(z_1)dz_1-2\,\mathrm{Re}\,U_2(z_2)=0 \quad \text{for} \quad 0\le r\le ct, \quad \theta=0$$

$$\frac{\sigma_{r\theta}}{\mu}=2\,\mathrm{Re}\int\frac{i(z_1^2-1)}{z_1^2+1}U_1'(z_1)dz_1+2\,\mathrm{Re}\int\left[\frac{i(z_2^2-1)}{z_2^2+1}-\frac{z_2^2+1}{i(z_2^2-1)}\right]U_2'(z_2)dz_2=0 \quad \text{for} \quad 0\le r\le at, \quad \theta=0 \tag{13}$$

$$\frac{\dot{u}_\theta}{a}=-\mathrm{Re}\int\frac{2i(z_1^2-1)z_1}{(z_1^2+1)^2}U_1'(z_1)dz_1+\mathrm{Re}\int\frac{2\bar{b}z_2}{i(z_2^2-1)}U_2'(z_2)dz_2=0 \quad \text{for} \quad ct\le r\le at, \quad \theta=0$$

Let us apply the conformal mapping methods using the functions

$$z_1=\frac{1-\sqrt{1-\xi_1^2}}{\xi_1} \quad \text{and} \quad z_2=\frac{\bar{b}-\sqrt{\bar{b}^2-\xi_2^2}}{\xi_2} \tag{14}$$

to map the upper parts of domains G_1 and G_2 in Fig. 3 onto domains Ω_1 and Ω_2 in the upper half-planes $\mathrm{Im}\,\xi_1>0$, $\mathrm{Im}\,\xi_2$

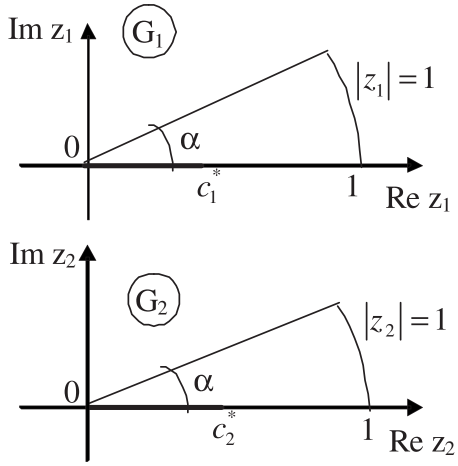

Fig. 3 Two domains in $z_{1,2}$ planes

>0 (Fig. 4). With such a choice of function (14), the values of different coordinates ξ_1 and ξ_2 coincide on the line OK (Fig. 4). In other words, domains Ω_1 and Ω_2 have a common boundary OK that also corresponds to the boundary on the X axis in Fig. 2, that is

$$\xi_1(z_1)=\xi_2(z_2)=\frac{x}{at} \quad (\theta=0,\ \mathrm{Im}\ \xi_1=\mathrm{Im}\ \xi_2=0) \tag{15}$$

Furthermore, under these mappings, the interval $[0,1]$ for $\theta=0$ on both G_1 and G_2 maps onto the intervals $[0,1]$ and $[0,\bar{b}]$ of the real axis of the plane $\xi_1=\xi_2=\xi$, respectively. In this plane, the second condition of formula (13) takes the form

$$\sigma_{r\theta}=2\ \mathrm{Re}\int \sqrt{\xi^2-1}\,U_1'(\xi)d\xi + \mathrm{Re}\int\left[\frac{\sqrt{\xi^2-\bar{b}^2}}{\bar{b}}-\frac{\bar{b}}{\sqrt{\xi^2-\bar{b}^2}}\right]U_2'(\xi)d\xi=0 \tag{16}$$

It is a well-known fact that longitudinal and transverse parts of stresses near the end of the crack both have the singularity of order 1/2 (e.g., see Ref. [6]). In this case, the lemma of Muskhelishvili [7] implies that either (1) the integrand in Eq. (16) should be discontinuous at the point $\xi=\bar{c}$ or (2) the whole integrand should be identically zero. Since the coefficients for $V_1'(\xi)$ and $V_2'(\xi)$ are regular functions, the integrand in Eq. (16) should be continuous at the point $\xi=\bar{c}$, implying that the whole integrand is zero for $0\le r\le at$, $\theta=0$, that is

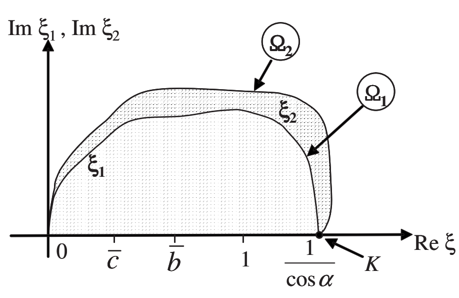

Fig. 4 Two domains in $\xi_{1,2}$ planes coincide on real axis

$$2\sqrt{\xi^2-1}\,U_1'(\xi)+\frac{\xi^2-2\bar{b}^2}{\bar{b}\sqrt{\xi^2-\bar{b}^2}}U_2'(\xi)=0 \tag{17}$$

Using Eq. (6), we can transform formula (17) to the form

$$V_1'(\xi)=\frac{\xi^2-2\bar{b}}{2\bar{b}^2}V_2'(\xi) \tag{18}$$

Thus, the main difficulty related to the availability of different argument functions is resolved.

Condition (18) enables us to separate different values of functions on a common boundary, and hence, to find the functions $V_1'(\xi)$ and $V_2'(\xi)$ separately. Namely, by virtue of Eq. (18), the first and the third conditions in Eq. (13) on the plane ξ take the form

$$\mathrm{Re}\int \frac{(\xi^2-2\bar{b})^2+4\bar{b}^3\sqrt{\xi^2-1}\sqrt{\xi^2-\bar{b}^2}}{2\bar{b}\sqrt{\xi^2-1}}{}^4V_2'(\xi)d\xi=0$$

$$\text{for}\quad \mathrm{Im}\ \xi=0,\quad 0\le \mathrm{Re}\ \xi<\bar{c}$$

$$\mathrm{Re}\int\left[\frac{(\xi^2-2\bar{b}^2)}{2\bar{b}^2}+1\right]V_2'(\xi)d\xi=0$$

$$\text{for}\quad \mathrm{Im}\ \xi=0,\quad \bar{c}<\mathrm{Re}\ \xi<\frac{1}{\cos\alpha} \tag{19}$$

By means of the mappings inverse to Eq. (14), we can rewrite condition (19) in the argument z_2 on plane G_2. These conditions, together with the remaining conditions on the parts of the boundary $|z_2|=1$ and on the line $\arg z_2=\alpha$, form a complete system for defining the unknown function $V_2'(z_2)$. In particular, conformally mapping domain G_2 onto the half-plane $\mathrm{Re}\ \omega_2>0$, we can reduce the problem to the known Riemann–Hilbert boundary value problem with discontinuous coefficient functions. Discontinuities hold at the origin of coordinates and at the end of the crack.

We provide the solution for this boundary value problem without details. On the line $\mathrm{Im}\ \omega_2=0$, we have

$$\mathrm{Im}\ V_2'(\omega_2)=0,\quad 0<\omega_2<\omega_{2c};$$

$$\mathrm{Re}\ V_2'(\omega_2)=0,\quad \omega_2<-\bar{b},\quad \omega_2>\omega_{2c}; \tag{20}$$

$$\mathrm{Re}\frac{z_2^2(\omega_2)V_2'(\omega_2)}{[1+z_2^2(\omega_2)]^2}=0,\quad -\bar{b}<\omega_2<0$$

where

$$\omega_{2c}=\frac{2\bar{b}c_*^n}{1+c_*^{2n}}$$

$$c_*=\frac{\bar{b}-\sqrt{\bar{b}^2-\bar{c}^2}}{\bar{c}}$$

$$\bar{c}=\frac{c}{a}$$

Introducing a new function

$$\Omega_2(\omega_2)=\frac{z_2^2(\omega_2)V_2'(\omega_2)}{[1+z_2^2(\omega_2)]^2}$$

we can reduce problem (20) to the form

$$\mathrm{Im}\ \Omega_2(\omega_2)=0\quad \text{for}\quad \mathrm{Im}\ \omega_2=0;\quad 0<\mathrm{Re}\ \omega_2<\omega_{2c}$$

$$\mathrm{Re}\ \Omega_2(\omega_2)=0\ \text{ for }\ \mathrm{Im}\ \omega_2=0;\quad \mathrm{Re}\ \omega_2<0;\quad \mathrm{Re}\ \omega_2>\omega_{2c} \tag{21}$$

that has an obvious solution for the considered case (see Ref. [7])

$$\Omega_2(\omega_2) = i(A + B\omega_2)\omega_2^{k/2}(\omega_2 - \omega_{2c})^{-3/2}$$

Here, k is an odd number chosen from the obvious convergence conditions at the end points $\omega_2=0$ and $\omega_2=\infty$; A and B are real numbers to be determined.

Simple calculations show that

$$\frac{z^2(\omega_2)}{1 + z_2^2(\omega_2)} = \begin{cases} O(\omega_2^{2/n}) & \text{for } \omega_2 \to 0 \\ O(1) & \text{for } \omega_2 \to \infty \end{cases} \tag{22}$$

from where we can define the number k as a solution of the following inequalities:

$$\left[\left(\frac{k}{2} + 1\right) - \frac{3}{2}\right] n \le 0 \Rightarrow k \le 1$$
$$\frac{kn}{2} - 2 > -1 \Rightarrow k > \frac{2}{n} \tag{23}$$

The only possible solution for these inequalities is if $n \ge 3$ and $k = 1$, which implies the following fundamental conclusion: Under the comprehensive stretching, there has to be at least three cracks; in particular, emergence of a single crack is deemed impossible.

As can be seen, the solution $V_2'(\omega_2)$ is determined to within a multiplier in the form of a polynomial $(A+B\omega_2)$, where real coefficients A and B can be determined from the conditions on the forefront. Thus, the desired solution is

$$V_2'(\omega_2) = (A + B\omega_2) i \frac{[1 + z_2^2(\omega_2)]}{z_2^2(\omega_2)} \omega_2^{1/2}(\omega_2 - \omega_{2c})^{-3/2} \tag{24}$$

The other unknown function $V_1'(z_1)$, may be determined on the basis of relation (18) that enables us to define boundary values of this analytical function on the edge $[0,1]$ in Fig. 3. This condition, together with the other boundary conditions on the remaining part of the domain G_1, completely define the function $V_1'(z_1)$ that depends on constants A and B.

Without going into too much detail, we provide the solution here. The function $V_1'(\xi_1)$ is a solution of the following boundary value problem on the plane ω_1, $z_1 = 1 - \sqrt{1-\omega_1^2}/\omega_1$, with boundary conditions on Im $\omega_1=0$ line as

$$V_1'(\omega_1) = f(\omega_1), \quad \omega_1 > 0$$

$$\mathrm{Im}\frac{z_1}{z_1^2 + 1} \cdot \frac{z_1}{i(z_1^2 - 1)} \cdot V_1'(\omega_1) = 0, \quad -1 < \omega_1 < 0$$

$$\mathrm{Re}\ V_1'(\omega_1) = 0, \quad \omega_1 < -1$$

where $f(\omega_1)$ is a known function, defined from Eq. (18). Solving this problem for V_1', we get

$$V_1'(\omega_1) = \frac{(z_1^2 + 1)}{z_1} \cdot \frac{i(z_1^2 - 1)}{z_1} \frac{\omega_1^{1/2}}{(\omega_1 - \omega_{1c})^{3/2}} \Omega(\omega_1) \tag{25}$$

where

$$\Omega(\omega_1) = \frac{1}{\pi}\int_0^1 \frac{\mathrm{Im}\,\tilde{f}(t)}{t - \omega_1} dt$$

with

$$\tilde{f}(\omega_1) = \frac{z_1}{(z_1^2 + 1)} \cdot \frac{z_1}{i(z_1^2 - 1)} \cdot \frac{(\omega_1 - \omega_{1c})^{3/2}}{\omega^{1/2}} f(\omega_1)$$

Observe that solutions (24) and (25) both contain unknown coefficients A and B that may be determined from the conditions on the fore front $\sigma_x = \sigma_0$ and $\sigma_y = \sigma_0$ on $|z_1| = 1$.

Calculating the corresponding integrals, we get

$$\frac{2\bar{b}^4\sigma_0}{\mu} = \frac{2\bar{b}^4}{\sqrt{1-\bar{b}^2}} \cdot \frac{[A + B\bar{\omega}_2(\bar{b})]\sqrt{\omega_2(\bar{b})}}{[\omega_2(\bar{b}) - \omega_2(\bar{c})]^{1/2}} + \int_{\bar{c}}^{\bar{b}} \frac{1}{[\omega_2(\xi) - \omega_2(c)]^{1/2}} \cdot \frac{d}{d\xi_2}\left[\frac{s(\xi)[A + B\omega_2(\xi)]\sqrt{\omega_2(\xi)}}{\sqrt{1-\xi^2}}\right] d\xi + \int_{\bar{b}}^{1} \frac{(\xi^2 - 2\bar{b}^2)[A + B\omega_2(\xi)]\sqrt{\omega_2(\xi)}\omega_2'(\xi)}{[\omega_2(\xi) - \omega_2(c)]^{3/2}\sqrt{1-\xi^2}} d\xi \tag{26}$$

$$\frac{2\bar{b}^4\sigma_0}{\mu} = \frac{-2\bar{b}^4(\bar{b}^2 - 3)\sqrt{\omega_2(\bar{b})}[A + B\omega_2(\bar{b})]}{\sqrt{1-\bar{b}^2}[\omega_2(\bar{b}) - \omega_2(\bar{c})]^{1/2}} + 2\int_{\bar{c}}^{\bar{b}} \frac{1}{[\omega_2(\xi) - \omega_2(c)]^{1/2}}\left[\frac{s_1(\xi)[A + B\omega_2(\xi)]\sqrt{\omega_2(\xi)}}{1 - \xi^2}\right] d\xi + \int_{\bar{b}}^{1} [(1 - 2\bar{b}^2)(\xi^2 - 1) + 1](\xi^2 - 2\bar{b}^2) \frac{[A + B\omega_2(\xi)]\sqrt{\omega_2(\xi)}\omega_2'(\xi)}{[\omega_2(\xi) - \omega_2(\bar{c})]^{3/2}\sqrt{1-\xi^2}} d\xi \tag{27}$$

where we denote

$$s(\xi) = (\xi^2 - 2\bar{b}^2) + 4\bar{b}^3\sqrt{\xi^2 - \bar{b}^2}\sqrt{\xi^2 - 1}$$

$$s_1(\xi) = [(1 - 2\bar{b}^2)(\xi^2 - 1) + 1](\xi^2 - 2\bar{b}^2) - 4\bar{b}^3\sqrt{\xi^2 - \bar{b}^2}\sqrt{\xi^2 - 1} \tag{28}$$

$$\omega_2(\xi) = \frac{2\bar{b}\left(\frac{\bar{b} - \sqrt{\bar{b}^2 - \xi^2}}{\xi}\right)^n}{1 + \left(\frac{\bar{b} - \sqrt{\bar{b} - \xi^2}}{\xi}\right)^{2n}}$$

Note that conditions (26) and (27) comprise a system of two linear equations in two unknowns A and B that can easily be solved to determine coefficients A and B, and hence, to arrive at the complete solution for V_1' and V_2'.

In particular, from these solutions using Eq. (9), one can get the solution for σ_θ, which in turn is used for determining an important stress field near the ends of the cracks—stress intensity coefficient, which after some algebra is found as

$$K_I = \lim_{\substack{s \to 0^+ \\ s = x - ct}} (\sigma_\theta \cdot \sqrt{2\pi s}) = \frac{2\sqrt{2\pi}\mu\sqrt{at}[(c^2 - 2\bar{b}^2)^2 - 4\bar{b}^3\sqrt{\bar{b}^2 - \bar{c}^2}\sqrt{1 - \bar{c}^2}][A + B(\omega_2(\bar{c}))]}{\sqrt{(1 - \bar{c}^2)\omega_2'(\bar{c})}} \tag{29}$$

Hence, we derived a very important characteristic of fracture—stress intensity coefficient near the end of the crack. The exact derivation of this coefficient is a fundamental breakthrough for this problem as it enables answering an important question of fracture mechanics—what will happen to the crack under given stress conditions. The answer to this question does not follow from the equations of continuum mechanics. Instead, the answer lies in formulating criteria for fracture, which is only made possible by expression (29).

4 Conclusion

In this work, the occurrence of star-shaped cracks at thin plates under comprehensive (uniform) stretching was experimentally confirmed, and the exact closed-form analytic solution to this problem was obtained. The numerous performed experiments cover different types of fracture, e.g., brittle, elastoplastic, etc. (Fig. 1), and all confirm our analytically proven fundamental conclusion: The number of cracks in thin plates under comprehensive stretching has to be at least *three*.

References

[1] Freund, L. B., 1990, *Dynamic Fracture Mechanics*, Cambridge University Press, Cambridge, UK.

[2] Broberg, K. B., 1960, "The Propagation of a Brittle Crack," Arkiv Fysik, **18**(2), pp. 159–192.

[3] Afanas'ev, E. F., and Cherepanov, G. P., 1973, "Some Dynamic Problems of the Theory of Elasticity," Appl. Math. Mech., **37**, pp. 584–606.

[4] Afanasy'ev, E. F., 1972, "Some Homogeneous Solutions of the Dynamic Theory of Elasticity," *Mexanika sploshnoy sredi i rodstvennie problemi analiza (Continium Mechanics and Related Problems of Analysis)*, N. I. Muskheleshvili, ed., Nauka, Moscow, pp. 31–39.

[5] Smirnov, V. I., 1974, *Kurs Visshey matematiki (Course of Higher Mathematics)*, Nauka, Moscow, Vol. 3.

[6] Parton, V. Z., and Boriskovsky, V. G., 1988, *Dynamics of Brittle Fracture*, Mashinostroenie, Moscow.

[7] Muskhelishvili, N. I., 1962, *Singulyarniye Integralniye Uravneniya (Singular Integral Equations)*, 2nd ed., FizMatGiz, Moscow.